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The Maslov class in a Riemannian phase space

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Abstract. *We investigate the relation between the Maslov class of a Lagrangian (Legendrian) submanifold, and its mean curvature vector.*

The Maslov class is a cohomological invariant of a pair of Lagrangian subbundles of a symplectic vector bundle, and its main applications are for Lagrangian submanifolds of a *phase space* which is the cotangent bundle T^*M of a manifold M. (Everything is C^∞ in this paper). In this Note we consider the case of a *Riemannian phase space* (T^*M, g), where g is a Riemannian metric of M, and we complete our computations of [11] such as to obtain a fully Riemannian expression of a differential form which represents the Maslov class. The result generalizes a formula given by J.M. Morvan [9] for the case $M = \mathbb{R}^n$. We shall also present a computation for the Maslov class of an *optical Lagrangian submanifold* [1] of T^*M. Furthermore, by a similar method, we give a Riemannian representative form of the Maslov class of a Legendrian submanifold of the unit sphere subbundle of T^*M. Finally, similar formulas are used to compute the mean curvature vector of a Lagrangian (Legendrian) submanifold of an almost Hermitian (almost contact metric) manifold. In the (almost) Hermitian case this yields an interpretation of a 1-form defined by A.T. Fomenko and Le Hong Van [7, 8].

Key Words: *Maslov class, Riemannian phase space, Lagrangian susubmanifold, Legendre submanifold, Mean curvature.*
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1. LAGRANGIAN SUBMANIFOLDS OF COTANGENT BUNDLES

Let $E \to V$ be a symplectic vector bundle of rank $2n$, and $L_{0,1}$ be Lagrangian subbundles of E. Then, after a preliminary reduction of the structure group of E to $U(n)$, $L_{0,1}$ yield further reductions to $O(n)$, and we may consider corresponding orthogonal connections $\theta_{0,1}$. The Maslov class $\mu(E, L_0, L_1) \in H^1(V, \mathbb{R})$ is the cohomology class of the 1-form $(\sqrt{-1}/\pi)(\theta^i_{1i} - \theta^i_{0i})$, where the components of the connections are with respect to unitary frames, and we agree to use the Einstein sum convention [5, 11].

Particularly, a phase space T^*M has the symplectic form $\Omega = d\lambda$, where λ is the Liouville 1-form, and the *vertical foliation* $\mathscr{V}$ by the fibers is Lagrangian. Then if L is a Lagrangian submanifold of T^*M, $\mu(TT^*M, \mathscr{V}/L, TL)$ is denoted by μ_L and called the Maslov class of L. It is the same as the class originally defined by Maslov in quantum physics. (In order to achieve this we added a factor -2 to the class considered in [11]).

In [11] a representative form of μ_L is given as follows. Put

$$(1.1)\qquad TT^*M = \mathscr{H} \oplus \mathscr{V},$$

where $\mathscr{H}$ is the *horizontal distribution* of the Riemannian connection D of g. Notice that $\mathscr{H} \approx \pi^{-1}(TM)$, and $\mathscr{V} \approx \pi^{-1}(T^*M)$ where $\pi: T^*M \to M$ is the projection, and the second isomorphism is Ω-duality. This allows to lift g to a Riemannian metric $\gamma = \pi^*g \oplus \pi^*g^*$ of T^*M, and to define the γ-metric connection $\hat{\nabla} = \pi^{-1}(D) \oplus \pi^{-1}(D^*)$ (g^*, D^* are g, D on T^*M). On the other hand, $J/\mathscr{H} = -(\Omega\text{-duality}) \circ (\gamma\text{-duality})$, $J^2 = -Id$ yields an almost complex structure on T^*M which is both g and Ω-compatible and satisfies $\hat{\nabla}J = 0$. Hence $\hat{\nabla}$ is a unitary connection, and it turns out that the Chern-Simons transgression form $T(\hat{\nabla})c_1$ restricted to unitary bases tangent to L (e.g., [11]) «lives» on L and it defines exactly $(1/2)\mu_L$.

We send the reader to [11] for more details and local coordinate expressions of this construction. Particularly, if (e_i, Je_i) are unitary bases along L such that $e_i \in TL$ $(i = 1, \ldots, n)$, and therefore Je_i are γ-normal to L, $\hat{\nabla}$ has local equations of the form

$$(1.2)\qquad \hat{\nabla}e_i = \lambda^j_i e_j + b^j_i(Je_j), \quad \hat{\nabla}(Je_j) = -b^j_i e_j + \lambda^j_i(Je_j),$$

where (λ^j_i), (b^j_i) are matrices of 1-forms which are antisymmetric and symmetric respectively. If $f_i = (e_i - \sqrt{-1}Je_i)/\sqrt{2}$, (1.2) gives $\hat{\nabla} f_i = (\lambda^j_i + \sqrt{-1}b^j_i)f_j$, hence $\hat{T}(\nabla)\, c_1 = -(1/2\pi)b^i_i$, and we get the following representative 1-form of μ_L [11]

$$(1.3)\qquad m_L = -(1/\pi)b^i_i$$

To go on we need the Riemannian connection ∇ of γ on T^*M, and we look for it under the form

$$(1.4) \qquad \nabla_X Y = \hat{\nabla}_X Y + S(X, Y).$$

If we write down the conditions that ∇ is γ-metric and torsionless, and use the classical cyclic permutation trick of Riemannian geometry we see that S is determined by the formula

$$(1.5) \qquad 2\gamma(S(X, Y), Z) = \hat{T}(X, Y, Z) - \hat{T}(Y, Z, X) - \hat{T}(Z, X, Y),$$

where $\hat{T}(X, Y, Z) = \gamma(X, \hat{T}(Y, Z))$, and $\hat{T}(Y, Z)$ is the torsion of $\hat{\nabla}$.

Remark. Formula (1.5) holds for any Riemannian manifold.

The torsion $\hat{T}$ was computed in [11] and an easy comparison shows that

$$(1.6) \qquad \hat{T}(X, Y, Z) = r(Z, Y, JX),$$

where r is a 3-covariant tensor defined on T^*M by

$$(1.7) \qquad r(X, Y, Z) = -\lambda\,(\text{horizontal lift of } R(\pi_* X, \ \pi_* Y)\,(\pi_* Z)),$$

and R is the Riemannian curvature tensor of (M, g).

THEOREM 1.1. *The Maslov class μ_L is the cohomology class of the form*

$$(1.8) \qquad m_L(X) = \frac{n}{\pi}\,(i(H_L)\Omega)(X) - \frac{1}{\pi}\,\Sigma_{i=1}^n\,(r\,(X, e_i, e_i) - r\,(JX, e_i, Je_i)),$$

where H_L is the mean curvature vector of L in (T^*M, γ), X is tangent to L and (e_i) is an arbitrary γ orthonormal basis of TL.

Proof. Since Je_i is a normal basis of L a known formula of Riemannian geometry yields

$$(1.9) \qquad nH_L = \Sigma_{i,j=1}^n\,\gamma(\nabla_{e_i} e_i, Je_j)\,Je_j,$$

and if we use (1.2), (1.4), and $b_i^j = b_j^i$ (1.9) becomes

$$(1.10) \qquad nH_L = \Sigma_{i,j=1}^n\,(\gamma(\hat{\nabla}_{e_i} e_i, Je_i) + \gamma(S(e_i, e_i), Je_j))\,Je_j.$$

Here, we shall replace $\hat{\nabla}_{e_i} e_j = \hat{\nabla}_{e_j} e_i + [e_i, e_j] + \hat{T}(e_i, e_j)$, and use (1.5) and (1.6). We get

$$nH_L = \Sigma_{i,j=1}^n b_i^i(e_j)Je_j - \Sigma_{i,j=1}^n (r(e_j, e_i, e_i) - r(Je_j, e_i, Je_i))Je_j.$$

Now, since (e_i, Je_i) is a symplectic basis, it is easy to check that this last result is equivalent to (1.8). Q.e.d. ∎

If $M = \mathbb{R}^n$, $r = 0$, and (1.8) reduces to Morvan's formula [9] The same holds if (M, g) is locally flat [11].

If (M, g) is of an arbitrary constant sectional curvature k, then $R(u, v)w = k\{g(v, w)u - g(u, w)v\}$ $(u, v, w \in TM)$, and (1.8) becomes

$$m_L(X) = \frac{n}{\pi}(i(H_L)\Omega)(X) + \frac{k}{\pi}\Sigma_{i=1}^n (g(\pi_* e_i, \pi_* e_i)\lambda(X) - g(\pi_* e_i, \pi_* Je_i)\lambda(JX) - g(\pi_* X, \pi_* e_i)\lambda(e_i) + g(\pi_* JX, \pi_* Je_i)\lambda(e_i)). \tag{1.11}$$

Then, if we decompose into *horizontal and vertical components*

$$X = {}^hX + {}^vX, \quad e_i = {}^he_i + {}^ve_i \tag{1.12}$$

and use $J(\mathscr{X}) = \mathscr{V}$, we obtain:

$$m_L(X) = \frac{n}{\pi}((i(H_L)\Omega)(X) - \frac{k}{\pi}\Sigma_{i-1}^n ([\gamma({}^hX, e_i) - \gamma({}^vX, e_i)]\lambda(e_i) + \gamma({}^he_i, {}^he_i)\lambda(X) + \Omega({}^he_i, {}^ve_i)\lambda(JX)). \tag{1.13}$$

For instance, if L is the conormal bundle ν^*N of a totally geodesic submanifold N of (M, g), it follows from our computations in [12] that, on one hand $m_L = 0$ and, on the other hand, TL has bases which consist of $p = \dim N$ horizontal vectors and of $n - p$ vertical vectors. This implies $\Omega({}^he_i, {}^ve_i) = 0$ and, since $\lambda = 0$ along a conormal bundle, formula (1.13) yields

PROPOSITION 1.2. *The conormal bundle ν^*N of a totally geodesic submanifold N of any space form M is a minimal Lagrangian submanifold of (T^*M, γ)* ∎

(If M is flat it suffices to ask only minimality of N in M [12]).

A Lagrangian submanifold L of T^*M is called *optical* [1] if it is a submanifold of a hypersurface W transversal to the fibers, and which intersects the latter along convex hypersurfaces. Here, we shall assume the stronger than transversality condition that TW does not contain the *Euler vector field E* defined by

$i(E)\,\Omega = \lambda$. In this case we call W a *regular hypersurface*, and if ι is the inclusion of W in T^*M $\eta = \iota^*\lambda$ is a contact form on W Indeed, it is easy to check using a basis of TW that

$$\eta \wedge (d\eta)^{n-1} = (1/n)\,\iota^*\,[i\,(E)\,(d\lambda)^n\,] \neq 0.$$

Let $C \in TW$ be defined by

$$i\,(C)\,(d\eta) = 0, \quad \gamma(C, C) = 1. \tag{1.14}$$

Then $\eta(C) \neq 0$, and we may orient C such as $\eta(C) > 0$. Moreover (1.14) implies that $\forall\, Z \in TW$ we have $\gamma(JC, Z) = \Omega(C, Z) = (i(C)d\eta)\,(Z) = 0$ i.e., JC is a unit normal vector field of W, hence it is not a horizontal vector, and C is not vertical. Accordingly, if C' is the orthogonal projection of JC onto $\mathscr{V}$ then $n = \nu C'$, $\nu = \|\, C' \,\|^{-1}$, is the unit normal vector of $W_x = W \cap \mathscr{V}_x$ in $(\mathscr{V}_x, \gamma)$ $(x \in M)$. Since it also follows from [11] that $\hat{\nabla}$ is the Riemannian connection of $(\mathscr{V}_x, \gamma)$, the second fundamental form of W_x in $\mathscr{V}_x$ is

$$\sigma(X, Y) = \gamma(\hat{\nabla}_X Y, n) = \nu\Omega(C, \hat{\nabla}_X Y) \qquad (X, Y \in TW_x).$$

Here, we can express $\hat{\nabla}$ by (1.4), and the presence of S will result in the presence of terms in r which vanish because $X, Y \in \mathscr{V}$. The result is

$$\sigma(X, Y) = \nu\Omega(C, \nabla_X Y) = \nu\gamma(JC, \nabla_X Y) = \nu\sigma_W(X, Y), \tag{1.15}$$

where σ_W in the second fundamental form of W in (T^*M, γ). Hence, we have

PROPOSITION 1.3. *The hypersurface W is convex along the fibers of T^*M iff $\mathscr{V} \cap TW$ is contained in the positive subspace of the second fundamental form of W in (T^*M, γ).* ■

Along the hypersurface W it is also important to consider the distribution $\mathscr{P}$ which is orthogonal to C, and, therefore it is Ω-orthogonal to the symplectic plane (C, JC). It follows that $(\mathscr{P}, \Omega/\mathscr{P})$ is a symplectic vector bundle on W.

Now, let $L \subset W$ be a Lagrangian submanifold of T^*M. The maximal Ω-isotropy of L implies that $C \in TL$, and that $\mathscr{L} = TL \cap \mathscr{P}$ is a Lagrangian subbundle of $(\mathscr{P}, \Omega/_{\mathscr{P}})$. Hence, in the computation of the Maslov class of L we may use local unitary bases $(e_\alpha, C, Je_\alpha, JC)$, where $\alpha = 1, \ldots, n-1$, and $e_\alpha \in \mathscr{L}$. With respect to these bases equations (1.2) become of the form

$$\begin{aligned} &\hat{\nabla} e_\alpha = \mu^\beta_\alpha e_\beta + \kappa_\alpha C + c^\beta_\alpha (Je_\beta) + \tau_\alpha (JC), \;\; \hat{\nabla}(Je_\alpha) = J\hat{\nabla} e_\alpha \\ &\hat{\nabla} C = -\Sigma^{n-1}_{\alpha=1} \kappa_\alpha e_\alpha + \Sigma^{n-1}_{\alpha=1} \tau_\alpha (Je_\alpha) + \theta(JC), \;\; \hat{\nabla}(JC) = J\hat{\nabla} C, \end{aligned} \tag{1.16}$$

where $\mu^\beta_\alpha = -\mu^\alpha_\beta$ and $c^\beta_\alpha = c^\alpha_\beta$. (Notice that $\hat{\nabla} C \perp C$ since $\gamma(C, C) = 1$).

Hence by (1.3) the Maslov class of L is represented by

(1.17) $$m_L = -\frac{1}{\pi} c^{\alpha}_{\alpha} - \frac{1}{\pi}\theta.$$

Now, we shall notice that since C is not vertical the orthogonal projection of $TW \cap \mathscr{V}$ onto $\mathscr{P}$ is a Lagrangian subbundle $\mathscr{L}_0$ of $(\mathscr{P}, \Omega/_{\mathscr{P}})$ such that a Maslov class $\mu(\mathscr{P}, \mathscr{L}_0, \mathscr{L})$ exists in $H^1(L, \mathbb{R})$. From (1.16) we see that $\hat{\nabla} f_{\alpha} = \mu^{\beta}_{\alpha} f_{\beta}$ and $\hat{\nabla} f_{\alpha} = (\mu^{\beta}_{\alpha} + \sqrt{-1}\, c^{\beta}_{\alpha}) f_{\beta}$, where $f_{\alpha} = (e_{\alpha} - \sqrt{-1}\, Je_{\alpha})/\sqrt{2}$, are $\mathscr{L}$ and $\mathscr{L}_0$ – orthogonal connections, respectively. This shows that $\mu(\mathscr{P}, \mathscr{L}_0, \mathscr{L})$ is represented precisely by the 1-form $m(\mathscr{L}_0, \mathscr{L}) = -(1/\pi)\, c^{\alpha}_{\alpha}$. Finally, the form θ of (1.17), which by (1.16) is $\theta(X) = \gamma(\hat{\nabla}_X C, JC)$ $(X \in TL)$, can be computed with (1.4), (1.5), (1.6) and (1.15). The result is

PROPOSITION 1.4. *For a Lagrangian submanifold L contained in a regular hypersurface W of T*M, the Maslov class* μ_L *is represented by the 1-form*

$$m_L(X) = m(\mathscr{L}_0, \mathscr{L})(X) - \frac{1}{\pi}\sigma_W(C, X) -$$

$$-\frac{1}{2\pi}(r(C, JC, JX) + r(X, C, C) + r(X, JC, JC)),$$

where notation was described previously. ∎

2. LEGENDRIAN SUBMANIFOLDS OF COTANGENT SPHERE BUNDLES

For the same manifold M as in Section 1, the cotangent sphere bundle is

(2.1) $$S^*M = \{p \in T^*M \,/\, g^*(p, p) = 1\},$$

and it is a regular hypersurface in the sense of Section 1. Indeed, since the Riemannian connection is length preserving its horizontal space $\mathscr{X}$ along S^*M is tangent to S^*M, and the latter must be transversal to the vertical distribution $\mathscr{V}$. Accordingly the normal vector of S^*M is in $\mathscr{V}$, and it is normal to $S^*M \cap \mathscr{V}$. Using natural local coordinates we see that this normal vector is exactly the Euler vector E, which proves our assertion.

Furthermore, it follows that C of Section 1 will be JE, and it is related to the contact form η induced by λ in S^*M by the relation

(2.2) $$\gamma(JE, X) = \Omega(E, X) = \lambda(X) = \eta(X),$$

therefore the distribution $\mathscr{P}$ of Section 1 is precisely the contact distribution $\eta = 0$. Accordingly, by definition, a *Legendrian submanifold* of S^*M (or T^*M)

is an $(n-1)$-dimensional submanifold Λ such that $T\Lambda$ are Lagrangian subspaces of this contact distribution. The Lagrangian distribution $\mathscr{L}_0$ encountered in the end of Section 1 is now exactly $TS^*M \cap \mathscr{V}$, and the Maslov class of Λ is defined as $\mu_\Lambda = \mu(\mathscr{P}, \mathscr{L}_0, T\Lambda)$.

As int the end of Section 1 (see also more details in [11]), we shall obtain a representative 1-form of μ_Λ from the connection $\hat{\nabla}$, by putting its local equations under the form (1.16), where $e_\alpha \in T\Lambda$, and C is to be replaced by JE (this also implies $\theta = -\gamma(\hat{\nabla}(JE), E) = \Omega(E, \hat{\nabla}E) = 0$ because $\hat{\nabla}$ preserves the vertical space which is Lagrangian). Hence the representative form of μ_Λ is [11]

$$(2.3) \qquad m_\Lambda = -\frac{1}{\pi} c_\alpha^\alpha.$$

Like in the Lagrangian case, in order to transform (2.3) into a Riemannian expression we shall compute the mean curvature vector H_Λ of Λ in S^*M. With the notation of formulas (1.16) the normal space of Λ in S^*M has the basis (Je_α, JE). Hence we shall have

$$(2.4) \qquad (n-1)H_\Lambda = \Sigma_{\alpha,\beta=1}^{n-1} \gamma(\nabla'_{e_\alpha} e_\alpha, Je_\beta)(Je_\beta) + \\ + \Sigma_{\alpha=1}^{n-1} \gamma(\nabla'_{e_\alpha} e_\alpha, JE)(JE),$$

where ∇' is the Riemannian connection of (S^*M, γ). Furthermore, from (1.16) and (1.4) we get

$$(2.5) \qquad \nabla'_{e_\alpha} e_\alpha = \hat{\nabla}_{e_\alpha} e_\alpha + \tau_\alpha E + S(e_\alpha, e_\alpha) - \gamma(S(e_\alpha, e_\alpha), E)E$$

and, therefore

$$(2.6) \qquad (n-1)H_\Lambda = \Sigma_{\alpha,\beta=1}^{n-1} c_\alpha^\beta(e_\alpha)(Je_\beta) + \\ + \Sigma_{\alpha,\beta=1}^{n-1} \gamma(S(e_\alpha, e_\alpha), Je_\beta)(Je_\beta) + \Sigma_{\alpha=1}^{n-1} \kappa_\alpha(e_\alpha)(JE) + \\ + \Sigma_{\alpha=1}^{n-1} \gamma(S(e_\alpha, e_\alpha), JE)(JE).$$

In this formula the terms containing S will be calculated by (1.5) and (1.6), and on the other hand we shall use

$$c_\alpha^\beta(e_\alpha) = c_\beta^\alpha(e_\alpha) = \gamma(\hat{\nabla}_{e_\alpha} e_\beta, Je_\alpha) = \\ = \gamma(\hat{\nabla}_{e_\beta} e_\alpha + [e_\alpha, e_\beta] + \hat{T}(e_\alpha, e_\beta), Je_\alpha) = \\ = c_\alpha^\alpha(e_\beta) + \gamma(\hat{T}(e_\alpha, e_\beta), Je_\alpha).$$

The final result is

$$(2.7) \qquad (n-1)H_\Lambda = \Sigma_{\alpha,\beta=1}^{n-1} c_\alpha^\alpha(e_\beta)(Je_\beta) - \Sigma_{\alpha,\beta=1}^{n} r(e_\beta, e_\alpha, e_\alpha)(Je_\beta) + \\ + \Sigma_{\alpha=1}^{n-1} \kappa_\alpha(e_\alpha)(JE) + \Sigma_{\alpha,\beta=1}^{n-1} r(Je_\beta, e_\alpha, Je_\alpha)(Je_\beta) +$$

$$+ \Sigma_{\alpha=1}^{n-1} r(JE, e_\alpha, Je_\alpha)(JE).$$

Now, since $(e_\alpha, JE, Je_\alpha, -E)$ is a symplectic basis Ω has the canonical form with respect to this basis, and we get from (2.3) and (2.7).

THEOREM 2.1. *The Maslov class of a Legendrian submanifold Λ of a cotangent sphere bundle S^*M is represented by the 1-form*

$$m_\Lambda(X) = \frac{n-1}{\pi}(i(H_\Lambda)d\eta)(X) - \tag{2.8}$$

$$-\frac{1}{\pi}\Sigma_{\alpha=1}^{n-1}\{r(X, e_\alpha, e_\alpha) - r(JX, e_\alpha, Je_\alpha)\}.$$

where $X \in T\Lambda$, η is the contact form of S^*M, and H_Λ is the mean curvature vector of Λ in (S^*M, γ). ■

The representative 1-form of the Maslov class of Λ given by (2.8) has obviously a Riemannian character in (S^*M, γ).

3. GENERALIZATION OF THE MEAN CURVATURE VECTOR FORMULAS

In this section we extend formulas (1.8) and (2.8) such as to compute the mean curvature vector of a Lagrangian submanifold of an almost Hermitian manifold and of a Legendrian submanifold of an almost contact metric manifold M. Of course, in this cases Lagrangian (Legendrian) means that the tangent bundle of the submanifold is a Lagrangian (Legendrian) subbundle with respect to the structure defined by the fundamental 2-form of the almost Hermitian (contact) structure. In the almost Hermitian case we reobtain a formula of Fomenko-Le Hong Van [7] and Le Hong Van [8], and we have a geometric interpretation of the 1-form defined by those authors.

Let (N^{2n}, g, J) be an almost Hermitian manifold, and $\Omega(X, Y) = g(JX, Y)$ be its fundamental (Kähler) 2-form. Let L be a Lagrangian submanifold on N. In view of the explanations given in Section 1, it is natural to define the *Maslov form* of L as

$$m_L = 2T(\hat{\nabla}/L)c_1 \tag{3.1}$$

where c_1 is the first Chern polynomial in diemnsion n, $\hat{\nabla}$ is the *Hermitian connection* of N (e.g., [6], Section IX.10), and T denotes the Chern-Simons transgression. In (3.1) the notation $\hat{\nabla}/L$ means that $\hat{\nabla}$ is restricted to the bundle of unitary frames (e_i, Je_i) $(i = 1, \ldots, n)$ of TN where $e_i \in TL$. This ensures that

m_L lives on L, but it may not be closed since, by [4], $dm_L = c_1$ (Curvature $\hat{\nabla}$) and $\hat{\nabla}$ is a unitary but, perhaps, not an orthogonal connection on the bundle mentioned above. (Notice also that $\hat{\nabla}$ of Section 1 is not the Hermitian connection of (T^*M, J, γ), but we have no analogue of that $\hat{\nabla}$ for a general almost Hermitian manifold).

Now, if we see (1.2) as being the local equations of the Hermitian connection $\hat{\nabla}/L$, where e_i are tangent to the submanifold L, (1.3) becomes the expression of the presently defined Maslov form (3.1) of L. Furthermore, we may use the formulas (1.4) and (1.5) in order to compute the Riemannian connection ∇ of g, and then compute like in the proof of Theorem 1.1. This will yield the formula:

$$(3.2) \qquad nH_L = \Sigma_{i,j=1}^n b_i^i(e_j)(Je_j) + \\ + \Sigma_{i,j=1}^n \{\hat{T}(Je_i, e_i, e_j) + \hat{T}(e_i, e_i, Je_j)\}(Je_j),$$

and, therefore, for every $X \in TL$, we shall have

PROPOSITION 3.1. *The mean curvature vector H_L of a Lagrangian submanifold L of an almost Hermitian manifold (N, J, g) is determined by the formula*

$$(3.3) \qquad n(i(H_L)\ \Omega)(X) = \pi m_L(X) + \\ + \Sigma_{i=1}^n \{\hat{T}(e_i, Je_i, X) + \hat{T}(Je_i, e_i, X)\},$$

(Notice that we made use of the following property of the Hermitian connection: $\hat{T}(Y, JZ) = \hat{T}(JY, Z)$) ([6], Proposition IX.10.2)). ■

Expressing the Nijenhuls tensor of J by the torsion of ∇ ([6], Proposition IX.3.6) we see that J is integrable iff

$$(3.4) \qquad \hat{T}(JX, JY) - J(\hat{T}(JX, Y)) - J(\hat{T}(X, JY)) - \hat{T}(X, Y) = 0,$$

and for the Hermitian connection this condition actually means

$$(3.5) \qquad \hat{T}(JX, JY) = J\hat{T}(X, JY) \quad \text{or} \quad \hat{T}(Z, JX, JY) = -\hat{T}(JZ, X, JY).$$

Accordingly, in the Hermitian case (3.3) becomes just nicely

$$(3.6) \qquad ni(H_L)\ \Omega = \pi m_L.$$

This is precisely the result of [7], with the supplementary information about the geometric nature of their ad hoc defined 1-form. Formula (3.3) is equivalent with the result obtained in a different manner in [8].

Similar formulas can be developed for Legendrian submanifolds of almost contact metric (a.c.m.) manifolds. Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be an almost contact metric manifold, where φ is the (1, 1)-tensor, ξ is the vector field, η is the 1-form, and g is the metric of the a.c.m. structure (e.g., [2]). Then it has a fundamental 2-form $\Phi(X, Y) = g(\varphi X, Y)$ which is nondegenerate on the vector bundle $im\varphi$, and, hence, it makes the latter into a symplectic vector bundle $\mathscr{C}$. If there exists a submanifold Λ of N such that $T\Lambda$ is a Lagrangian subbundle of $\mathscr{C}$ then Λ will be called a Legendrian submanifold of N

An a.c.m. manifold has an important connection $\hat{\nabla}'$ which may be defined as follows. Consider the well known almost Hermitian structure on $N \times \mathbb{R}$ defined by the tensors

$$(3.7)\qquad J\left(X \oplus a\,\frac{d}{dt}\right) = \left((\varphi X - a\xi) \oplus \eta(X)\,\frac{d}{dt}\right),$$

$$\gamma\left(X \oplus a\,\frac{d}{dt},\quad Y \oplus b\,\frac{d}{dt}\right) = g(X, Y) + ab.$$

Then, define $\hat{\nabla}$ to be the Hermitian connection of $N \times \mathbb{R}$, and $\hat{\nabla}'$to be induced by $\hat{\nabla}$ in $N \times (0) = N$. Then $\hat{\nabla}(\hat{\nabla}')$ induces a unitary connection $\mathring{\nabla}$ in $(\mathscr{C}, \varphi, g)$, and we shall define the *Maslov form* of the Legendrian submanifold Λ on N by means of the transgression form

$$(3.8)\qquad m_\Lambda = 2T(\mathring{\nabla}/_\Lambda)\,c_1\,.$$

In order to compute this form, we shall represent the Hermitian connection of $N \times \mathbb{R}$ by equations of the form (1.16), where C is to be replaced by ξ and JC by d/dt, and where $e_\alpha \in T\Lambda$. If we also replace the Greek indices by Latin indices since they have to run from 1 to n, we get like for (2.3)

$$(3.9)\qquad m_\Lambda = -\frac{1}{\pi}\,c^i_i.$$

Furthermore, we may compute again like for the proof of the formulas (2.7), (2.8), and thereby obtain for the mean curvature vector H_Λ the result

$$(3.10)\qquad nH_\Lambda = \Sigma^n_{i,j=1} c^i_i(e_j)\,(Je_j) + \Sigma^n_{i=1}\,\kappa_i(e_i)\xi +$$
$$+ \Sigma^n_{i,j=1}[\hat{T}(Je_i, e_i, e_j) + \hat{T}(e_i, e_i, Je_j)]\,(Je_j) + \Sigma^n_{i=1}\hat{T}(e_i, e_i, \xi)\xi.$$

Now, if X is a tangent vector field of Λ, we get

PROPOSITION 3.2. *The mean curvature vector H_Λ of a Legendrian submanifold Λ of the a.c.m. manifold N is determined by the formulas*

$$(3.11)\qquad n(i\,(H_\Lambda)\Phi)\,(X) = \pi m_\Lambda\,(X) - \Sigma^n_{i=1}(\hat{T}(\varphi e_i, e_i, X) + \hat{T}(e_i, \varphi e_i, X)),$$

$$(3.12) \qquad n\eta(H_\Lambda) = \Sigma_{i=1}^n \kappa_i(e_i) + \Sigma_{i=1}^n T(e_i, e_i, \xi) = = -\Sigma_{i=1}^n g(e_i, [e_i, \xi]).$$

The last equality follows using the formulas (1.16) of the present case. Let us also remember that if J of (3.7) is integrable the a.c.m. structure is said to be *normal*, and then, just like for (3.6), we get

$$(3.13) \qquad n(i(H_\Lambda)\Phi) = \pi m_\Lambda .$$

Let us recall again the basic formula [4]

$$(3.14) \qquad d(T(\hat{\nabla})\, c_1) = c_1(\hat{\Omega}),$$

where $\hat{\Omega}$ is the curvature of $\hat{\nabla}$ and $T(\hat{\nabla})c_1$ is a form on the total space of the corresponding bundle of unitary frames. In view of the formulas established earlier in this Section (3.14) indicates a relation between the minimality property of Lagrangian (Legendrian) submanifolds and the first Chern class. Namely we have

PROPOSITION 3.3. *Let L be a minimal Lagrangian submanifold of a Hermitian manifold M. Then the first Chern class $c_1(M)$ vanishes on L. Similarly, if Λ is a minimal Legendrian submanifold of a normal almost contact manifold, then $c_1(\mathscr{C})$ vanishes on Λ.*

Proof. The first part of this Proposition was established directly in [3] and [8]. It follows from (3.6), and the transgression interpretation of m_L by restricting (3.14) to L. The second part follows in the same way from (3.13) ■

REMARK. *The results of Proposition 3.3 remain valid for the almost Hermitian and contact case if T vanishes along the submanifolds L and Λ respectively.*

Relations between the first Chern class and stability of minimal Lagrangian submanifold were established straightforwardly in [8] and [10] by a study of the second variation of the volume. For instance, if the first Chern class of a Hermitian manifold is negative its minimal Lagrangian submanifolds are stable [8].

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